M303 Remote exam practice paper.

Please note that this paper has not gone through our usual checking processes and so may contain mistakes and/or omissions.

Part 1

You should **attempt all questions**.
Each question is worth 11%.

Question 1

(a) Let X_n be the sequence defined by the following recurrence.

$$X_1 = 1, X_2 = 1$$
$$X_{n+1} = X_n + X_{n-1} \qquad \text{for } n \geq 2.$$

Use the Second Principle of Mathematical Induction to prove that $X_{n+5} = X_4X_n + X_5X_{n+1}$ [6]

Solution

First we write down a few terms.

$$X_1 = 1,\ X_2 = 1,\ X_3 = 2,\ , X_4 = 3\ X_5 = 5,\ X_6 = 8,\ X_7 = 13.$$

Let $P(n)$ be the proposition:

$$X_{n+5} = X_4X_n + X_5X_{n+1}.$$

We check the base cases. $P(1)$ states that $X_6 = X_4X_1 + X_5X_2$. Since $8 = 3 \times 1 + 5 \times 1$ we have $P(1)$ is true.

$P(2)$ states that $X_7 = X_4X_2 + X_5X_3$. Since $13 = 3 \times 1 + 5 \times 2$ we have $P(2)$ is true.

Assume that $P(n)$ is true for $n = 1, \ldots, k$, and consider $P(k+1)$.

$P(k+1)$ is the proposition:

$$X_{k+6} = X_4X_{k+1} + X_5X_{k+2}.$$

By the assumption, (and because we have checked 2 base cases), we can assume that $P(k)$ and $P(k-1)$ hold. Ie:

$$X_{k+5} = X_4X_k + X_5X_{k+1}$$

and

$$X_{k+4} = X_4X_{k-1} + X_5X_k.$$

Adding these together gives:

$$X_{k+5}+X_{k+4} = X_4X_k+X_5X_{k+1}+X_4X_{k-1}+X_5X_k = X_4(X_k+X_{k-1})+X_5(X_{k+1}+X_k),$$

but, by definition of X_n we have $X_{k+5}+X_{k+4} = X_{k+6}$, $X_k+X_{k-1} = X_{k+1}$ and $X_{k+1} + X_k = X_{k+2}$. Substituting gives:

$$X_{k+6} = X_4X_{k+1} + X_5X_{k+2}$$

as required. Therefore by the Second Principle of Mathematical Induction $P(n)$ is true for all $n \geq 1$.

(b) From the definition of congruence alone, prove the following, for integers a, b, c, k and n with $n > 1$ and $k > 0$. If $a \equiv b \pmod{n}$ then $ka + c \equiv kb + c \pmod{n}$. [2]

Solution

Since $a \equiv b \pmod{n}$ we have $a - b$ divisible by n (Defn 1.1 HB p18) and so $a-b = rn$ for some r. RTP $ka+c-(kb+c)$ is divisible by n. However $ka + c - (kb + c) = k(a - b) = krn$ and so (again by Defn 1.1 HB p18) $ka + c \equiv kb + c \pmod{n}$. 2

(c) Suppose that m and n are positive integers such that n has remainder 3 when divided by 4 and that $m = 9n + 4$. Prove that m and n are coprime. [3]

Solution

Since n has remainder 3 when divided by 4 we have $n = 4k + 3$ for some k. Therefore $m = 9(4k + 3) + 4 = 36k + 31$. So we want the hcf of $4k + 3$ and $36k + 31$. Assume r divides both, then r divides $36k + 31 - 9(4k + 3)$ (Thm 4.4(e) HB p15) so $r|4$ and $r = 1, 2$ or 4. But $r|n$ and n is odd, so r must be odd and so $r = 1$. 3

Question 2

(a) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_{11}$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(0, 3)$. [1]

(iii) Write down an element of order 22. [1]

(iv) Is G cyclic? Justify your answer. [1]

Solution

Let $G = \mathbb{Z}_4 \times \mathbb{Z}_{11}$.

(i) 44

(ii) 11

(iii) $(2, p)$ where $p < 11$ and $p \neq 0$

(iv) yes (Corollary 2.12 - 4 and 11 are coprime) or Proposition 2.10 (an element of order 44). 4

(b) Consider the dihedral group D_{22} given by

$$D_{22} = \langle r, s | r^{22} = s^2 = e, sr = r^{21}s \rangle.$$

(i) Write the element $r^{11}sr^{11}$ in the form $r^i s^j$ for some $i \in \{0, \ldots, 21\}$ and some $j \in \{0, 1\}$.

(ii) Find the order of the element $r^{11}s$.

(iii) Is the subgroup $\langle r^{11} \rangle$ normal in D_{22}? Justify your answer. [4]

Solution

Consider the dihedral group D_{22}

(i) $r^{11} = r^{-11}$ so using relation in HB page 34 ($sr^{-i} = r^i s$) we have $r^{11}sr^{11} = r^{11}sr^{-11} = r^{11}r^{11}s = s$

(ii) $r^{11}sr^{11}s = ss = e$. $sr^{11} = e$ implies that $s = r^{11}$ which is impossible since D_{22} is not abelian. Therefore order 2.

(iii) yes, r^{11} is in the centre and so the subgroup must be normal since $gr^{11}g^{-1} = r^{11}$ for all $g \in G$.

4

8 Total

(c) Let f be an onto homomorphism from a group G to a group H. Prove that if G is abelian then so is H. [3]

Solution

Consider $a, b \in H$. Since f is onto we can find $h, k \in G$ with $f(h) = a$ and $f(k) = b$. Since f is a homomorphism we have $ab = f(h)f(k) = f(hk) = f(kh) = f(k)f(h) = ba$ and so H is abelian.

Question 3

(a) Given $28 = 2^2 \times 7$, list all the divisors of 28.

Hence, or otherwise, find

(i) $\tau(28)$

(ii) $\sigma(28)$

(iii) $\phi(28)$ [3]

Solution

The divisors are $1, 2, 4, 7, 14, 28$.

(i) $\tau(28) = 6$. Defn 2.2 of C2, HB pg 17 or 1.1 of C9 HB pg 42.

(ii) $\sigma(28) = 1 + 2 + 4 + 7 + 14 + 28 = 56$. Defn 1.2 of C9, HB pg 42.

(iii) $\phi(28) = 1 \cdot 2 \cdot 6 = 12$.
Defn 3.1 of C9, HB pg 43. Prop 2.2 of C9, HB pg 44
or 1,3,5,9,11,13,15,17,19,23,25,27
References not needed for marks.

3

3 Subtotal (a)

(b) Show that $x^4 + 8$ is irreducible over $\mathbb{Z}$. [4]

Solution

First assume that x^4+8 has a linear factor. Then it has a root in $\mathbb{Z}$ by thm 4.2 HB p 51. However if $x^4 = -8$ then $(x^2)^2 = -8$ which is impossible. Therefore there are no linear factors and the only possible factorisation is $x^4 + 8 = (ax^2 + bx + c)(px^2 + qx + r)$. Comparing coefficients: $ap = 1$ so a and p are multiplicative inverses (ie each +1 or each -1) and so by multiplying $ax^2 + bx + c$ by p and the other polynomial by q if necessary we may assume $a = p = 1$.

$x^4 + 8 = (x^2 + bx + c)(x^2 + qx + r)$.

now comparing coefficients of x^3 gives $b + q = 0$ so $b = -q$ giving:

$x^4 + 8 = (x^2 + bx + c)(x^2 - bx + r)$.

Now compare coefficients of x gives $rb - cb = b(r - c) = 0$. Since $\mathbb{Z}$ is an integral domain we have either $b = 0$ or $r = c$.

First assume $b = 0$. Then we have $x^4+8 = (x^2+c)(x^2+r)$ and comparing coefficients of x^2 now gives $r = -c$ and so $-c^2 = 8$ (which has no real solution).

Now assume $r = c$. This gives:

$x^4 + 8 = (x^2 + bx + c)(x^2 - bx + c)$. Now considering the constant term gives $c^2 = 8$ which has no solution in $\mathbb{Z}$ and so $x^4 + 8$ is irreducible over $\mathbb{Z}$.

(c) Use the rational root test to show that

$$f(x) = x^3 - 4x^2 + 8x + 2$$

is irreducible over $\mathbb{Q}$. [4]

Solution

By the rational root test, any root $r = \frac{p}{q}$ must have p dividing the constant term, and q dividing the leading coefficient. Thus, we have $p = \pm 1, \pm 2$, and $q = \pm 1$, which gives four possibilities for r:

$$\begin{aligned} r = 1: &\quad f(1) = 1 - 4 + 8 + 2 = 7 \neq 0 \\ r = -1: &\quad f(-1) = -1 - 4 - 8 + 2 = -11 \neq 0 \\ r = 2: &\quad f(2) = 8 - 16 + 16 + 2 = 10 \neq 0 \\ r = -2: &\quad f(-2) = -8 - 16 - 16 + 2 = -38 \neq 0. \end{aligned}$$

Therefore $f(x)$ has no linear factor, and so (since the degree of $f(x)$ is 3) f must be irreducible over $\mathbb{Q}$.

4

4 Subtotal (c)

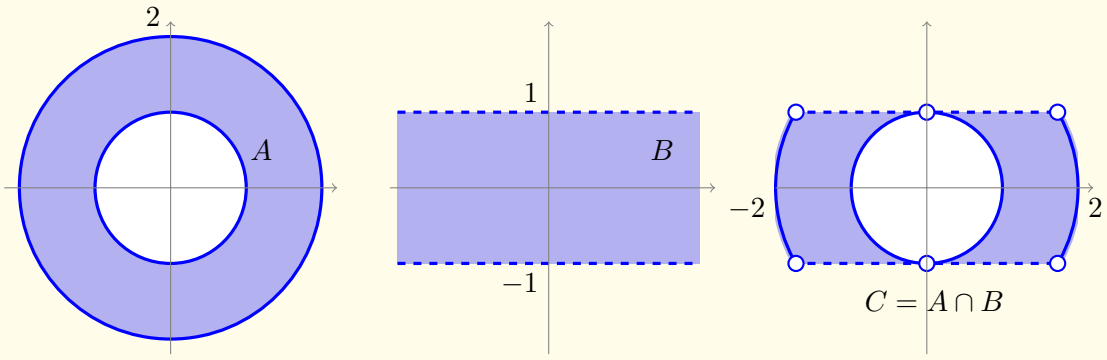
Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4\},$$
$$B = \{(a, b) \in \mathbb{R}^2 : |b| < 1\} \quad \text{and}$$
$$C = A \cap B.$$

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

Solution



1 each set

Sketches need only be qualitatively correct and give right status to boundary points but do need to give exact coordinates (where relevant) for full marks.

Description: the first sketch (labelled A) shows a shaded annulus, with solid edges. It is composed of two circles, both centred on the origin. The inner circle has radius 1 and the outer circle has radius 2.

The second sketch (labelled B) represents an 'infinitely long' horizontal strip with dotted, straight line, horizontal, edges top and bottom. The upper horizontal line goes through the point $(0, 1)$ and the lower horizontal line goes through the point $(0, -1)$.

The final sketch (labelled $C = A \cap B$) shows a shape that looks similar to a rectangle, but with horizontal top and bottom edges and curved sides. The left hand side is a solid circular arc (part of a circle centred at the origin) from $(-\sqrt{3}, 1)$ to $(-\sqrt{3}, -1)$. The right hand side is a solid circular arc (part of a circle centred at the origin) from $(\sqrt{3}, 1)$ to $(\sqrt{3}, -1)$. The top horizontal edge, from $(-\sqrt{3}, 1)$ to $(\sqrt{3}, 1)$, and the bottom horizontal edge, from $(-\sqrt{3}, -1)$ to $(\sqrt{3}, -1)$, are both dotted. The four 'corners' are open circles. Inside this shaded shape there is a missing disc, of radius 1, with solid edges. There are also open circles on the points $(0, 1)$ and $(0, -1)$.

(b) Write down the closure, interior and boundary of C for the Euclidean metric. [3]

Solution

The question states 'write down', so we do not need to justify the answers.

$$\mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) = \{(a,b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4 \text{ and } |b| < 1\}$$

$$\mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C) = \{(a,b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 4 \text{ and } |b| < 1\}$$

$$\mathrm{Bd}_{(\mathbb{R}^2,d^{(2)})}(C) = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 \in \{1,4\} \text{ and } |b| \leq 1\}$$
$$\cup \{(a,b) \in \mathbb{R}^2 : a^2 \leq 3 \text{ and } b \in \{-1,1\}\}.$$

3

(c) Write down the closure and interior of C for the mixed metric, e, given by:

$e\big((x_1,x_2),(y_1,y_2)\big) = d_0(x_1,y_1) + |x_2 - y_2|$ for $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2$. [2]

Solution

The question states 'write down', so we do not need to justify the answers.

$$\mathrm{Cl}_{(\mathbb{R}^2,e)}(C) = \{(a,b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4 \text{ and } |b| \leq 1\}$$

$$\mathrm{Int}_{(\mathbb{R}^2,e)}(C) = \{(a,b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 4 \text{ and } |b| < 1\}$$

2

(d) Determine the e-boundary of $S = \{(0,b) : 0 < b \leq 1\}$ in $\mathbb{R}^2$. [3]

Solution

By Theorem 5.4 of Chapter 16 (HB p86) the e-boundary of S is equal to the set of e-boundary points, that is the set of all points (a,b) that are such that $B_e((a,b),r)$ meets S and S^c for all r.

For any point $\big(0,b\big) \in S - \{(0,1)\}$, $B_e(\big(0,b\big),r) \subseteq S$ for $r < \min\{1-b,b\}$ and so none of these points can be boundary points.

For any r we can find an $n \in \mathbb{N}$ such that $\frac{1}{n} < r$ and so $B_e((0,0),r)$ contains $(0,0) \in S^c$ and $(0,\frac{1}{n}) \in S$. Therefore $(0,0)$ is a boundary point.

Also $B_e((0,1),r)$ contains $(0,1+\frac{r}{2}) \notin S$ and $(0,1) \in S$ for any r. Therefore $(0,1)$ is a boundary point.

For any point (a,b) with $a \neq 0$, $B_e((a,b),r)$ does not meet S for $r < 1$.

For any point $(0,b)$ with $b < 0$ $B_e((0,b),\frac{b}{2})$ does not meet S and for any point $(0,b)$ with $b > 1$ $B_e((0,b),\frac{b-1}{2})$ does not meet S.

Therefore the boundary of S is $\{(0,1),(0,1)\}$.

3

Question 5

Let $\alpha = \sqrt{11} - \sqrt{3}$; you may assume that
$[\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, $\sqrt{11} \notin \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \notin \mathbb{Q}(\sqrt{11})$.

(a) Express α^3 in the form $a\sqrt{11} + b\sqrt{3}$ for suitable $a, b \in \mathbb{Q}$. [*2*]

(b) With the help of (a) or otherwise, show that $\sqrt{11}, \sqrt{3} \in \mathbb{Q}(\alpha)$. [*3*]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$. [*1*]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [*2*]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [*3*]

Solution

(a) We have $\alpha^3 = (\sqrt{11})^3 - 3(\sqrt{11})^2\sqrt{3} + 3\sqrt{11}(\sqrt{3})^2 - (\sqrt{3})^3$ 2
and so $\alpha^3 = 11\sqrt{11} - 3 \cdot 11 \cdot \sqrt{3} + 3 \cdot \sqrt{11} \cdot 3 - 3\sqrt{3} = 20\sqrt{11} - 36\sqrt{3}$.

(b) Eliminating $\sqrt{11}$ and $\sqrt{3}$ from α and substituting into the expression for α^3 gives
$\alpha^3 - 20\alpha = -16\sqrt{3}$, and
$\alpha^3 - 36\alpha = -16\sqrt{11}$.
But both $\alpha^3 - 20\alpha$ and $\alpha^3 - 36\alpha$ are in $\mathbb{Q}(\alpha)$ and so $\sqrt{11}, \sqrt{3} \in \mathbb{Q}(\alpha)$. 3

(c) By (b) one has $\mathbb{Q}(\sqrt{11}, \sqrt{3}) \subseteq \mathbb{Q}(\alpha)$,
and since $\alpha \in \mathbb{Q}(\sqrt{11}, \sqrt{3})$ it follows that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$. 1

(d) Since $\sqrt{11} \notin \mathbb{Q}(\sqrt{3})$ and $\sqrt{11}$ is a root of a quadratic polynomial over $\mathbb{Q}(\sqrt{3})$ (in fact, over $\mathbb{Q}$ – also acceptable), we have $[\mathbb{Q}(\sqrt{11}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$.
By the KLM Theorem (Theorem 2.29 of Chapter 18) and by (c) we have $[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{11}, \sqrt{3}) : \mathbb{Q}(\sqrt{3})] \cdot [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2 \cdot 2 = 4$. 2
(The roles of $\sqrt{11}$ and $\sqrt{3}$ are, of course, interchangeable in the above argument.)

(e) Squaring α gives $\alpha^2 = 14 - 2\sqrt{11 \cdot 3}$, and hence $\alpha^2 - 14 = -2\sqrt{11 \cdot 3}$.
Squaring again we obtain $\alpha^4 - 28\alpha^2 + 14^2 = 4 \cdot 11 \cdot 3 = 132$, that is, $\alpha^4 - 28\alpha^2 + 64 = 0$
Since by (d) the minimal polynomial for α over $\mathbb{Q}$ must have degree 4, the sought minimal polynomial is $x^4 - 28x^2 + 64$. 3

11 Total

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(a, b) : b = 0\}$ by

$$d(\mathbf{x}, \mathbf{y}) = 3|x_1 - y_1| + |x_2 - y_2|$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(a) Write down $d\big((-1, 1), (1, 1)\big)$. [*1*]

Solution

$3 \times 2 + 0 = 6$. **1 Subtotal**

(b) Write down a d-disconnection of X. [*2*]

Solution

$\{\{(a, b) : b > 0\}, \{(a, b) : b < 0\}\}$. 2

(c) (i) Show that $\big((0, 1/n)\big)$ is not a d-convergent sequence in X. [*4*]

Solution

Let $(a, b) \in X$, then $b \neq 0$.
Hence for $n \in \mathbb{N}$,

$$\begin{aligned} d((0, 1/n), (a, b)) &= 3|a| + |1/n - b| \\ &\geq |b| - 1/n, \end{aligned}$$

by the reverse triangle inequality.
Hence if $N \in \mathbb{N}$ is chosen so that $1/N < |b|/2$, then for $n > N$, we have

$$d((0, 1/n), (a, b)) > |b|/2 > 0, \quad 2$$

and so $(d((0, 1/n), (a, b)))$ is not a real null sequence and $((0, 1/n))$ does not d-converge to (a, b).
Since (a, b) is an arbitrary point in X, we conclude that $((0, 1/n))$ is not d-convergent in X. 4

Alternatively
Let e be the metric on $\mathbb{R}^2$ defined by

$$e(\mathbf{x}, \mathbf{y}) = 3|x_1 - y_1| + |x_2 - y_2|.$$

Then $e((0, 1/n), (0, 0)) = |1/n|$ and so $(e((0, 1/n), (0, 0)))$ is a null sequence and $((0, 1/n))$ e-converges to $(0, 0)$. However $d = e_X$ so by Theorem 1.5 of Ch15, if $(0, 1/n)$ is d-convergent to $(p, q) \in X$ then it is also e-convergent to (p, q), contradicting Thm 3.3 of Ch14 (limits unique in metric spaces).

(ii) Hence, or otherwise, show that X is not d-complete. [3]

Solution

Each term of the sequence $((0, 1/n))$ lies in X.

If we show that it is a Cauchy sequence, then the result of part (ii) immediately implies that the space is not complete.

Let $\mathbf{a}_n = (0, \frac{1}{n})$ for $n \in \mathbb{N}$. Then for $n < m \in \mathbb{N}$,

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a}_m)) &= |1/n - 1/m| + 0 \\ &\leq 1/n. \end{aligned}$$

Hence for $\epsilon > 0$, choose N so that $1/N < \epsilon$ and then for $m > n > N$, $d(\mathbf{a}_n, \mathbf{a}_m) < \epsilon$ 3

and so the sequence is Cauchy.

(iii) Deduce that X is not d-sequentially compact. [1]

Solution

If X was sequentially compact, then Theorem 2.3 of Chapter 23 (HB page 136) and Theorem 5.12 of Chapter 22 (HB page 134) would imply that X was complete, which it isn't. (Or sequence above does not have a convergent subsequence.) 2 1

Part 2

You should **attempt two questions**. If you attempt more, only the scores from your first two questions will count.
Each question is worth 17%.

Question 7

(a) Let G be a group of order $2116 = 2^2 \times 23^2$

(i) Write down the order of the Sylow 23-subgroups and the Sylow 2-subgroups. [*1*]

(ii) Determine the possible numbers of Sylow 2-subgroups and Sylow 23-subgroups. [*4*]

Solution

(i) 4 and $23^2 = 529$.

(ii) Let n_2 be the number of Sylow 2-subgroups. Then by Sylow's second thm (1.12 HB p39) $n \equiv 1 \bmod 2$ and by Sylow's third thm part b (1.12 HB p39) $m|23^2$.
Therefore $n = 1$ or 23 or 529.
Let m be the number of Sylow 23-subgroups. Then by Sylow's second thm (1.12 HB p39) $m \equiv 1 \bmod 23$ and by Sylow's third thm part b (1.12 HB p39) $m|4$.
Therefore $m = 1$. 4

(b) Let G be a non-abelian group of order 2116.

Using part (a), prove that G cannot have a unique Sylow 2-subgroup. [*4*]

Solution

Assume G has a unique Sylow 2-subgroup, L and let K be the unique Sylow 23-subgroup. Then $|L| = 4$ and $|K| = 23^2$ so L and K are both abelian by Thm 2.15 of Chapter 7, HB p36. Since both are unique both are normal by Lemma 1.9 of Chapter 8 HB p39. We note that $|L|$ and $|K|$ are coprime and their product is $|G|$. Therefore by Thm 2.3 (HB p39) G would be an internal direct product of Abelian groups and so abelian. It isn't. 4

(c) Let H be the dicyclic group of order 2116

$$\mathrm{Dic}_{529} = \langle a, b | a^4 = e, a^2 = b^{529}, aba^3b = e \rangle.$$

and let K be the dihedral group of order 2116

$$D_{1058} = \langle r, s | r^{1058} = s^2 = e, sr = r^{1057}s \rangle$$

(i) Write down a group isomorphic to the Sylow 2-subgroup of H.

(ii) Write down a group isomorphic to the Sylow 2-subgroup of K.

(iii) Prove that the centre of H has an element of order 2.

(iv) Determine all elements in the centre of K. [*5*]

Solution

(i) $\mathbb{Z}_4$.

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) The element $a^2 = b^{529}$ is a power of both a and b so clearly commutes with both and so is in the centre of H. Since $a^4 = e$ and $a^2 \neq e$ it has order 2.

(iv) Assume that an element of the form $r^i \in Z(K)$. Then this element must commute with s.
Now $sr^i = r^i s$ if, and only if $sr^i = r^{-i}s$ by note on HB p34. This happens if, and only if, $r^i s = r^{-i}s$ which happens if, and only if $r^i = r^{-i}$ which is true if, and only if, $r^{2i} = e$. Therefore the only candidate for an element in the centre of this form is r^{529}.
This clearly commutes with r so it is in the centre.
All other elements of K are of the form $r^i s$ (the reflections). Assume that an element of this form is in the centre of K. This element must commute with s and so we have:
$r^i s \in Z(K)$ if, and only if $sr^i s = r^i ss = r^i$. However $sr^i s = ssr^{-i} = r^{-i}$ so $r^i s \in Z(K)$ if, and only if $r^i = r^{-i}$ and again this happens if, and only if, $r^{2i} = e$. Therefore the only possible central element of this form is $r^{529}s$. To be in the centre this would also need to commute with r. Assume (for a contradiction) that $r^{530}s = r^{529}sr$. This implies that $r^{530}s = r^{529}r^{1057}s = r^{1586}s = r^{528}s$. Clearly $r^{530}s \neq r^{528}s$ Therefore we have shown by contradiction that $r^{529}s$ is not in the centre of K and $Z(K) = \{e, r^{529}\}$. 5

(d) Prove that $\mathbb{R}/\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ cannot be isomorphic. [3]

Solution

Assume (for a contradiction) that we have an isomorphism from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Q}$.

The elements of $\mathbb{R}/\mathbb{Q}$ are cosets of the form $s+\mathbb{Q}$ for some $s \in \mathbb{R}$. Choose $r \in \mathbb{R}$ such that r is transcendental. Then since no power of $r \in \mathbb{Q}$ we have that $r + \mathbb{Q}$ has infinite order. The elements of $\mathbb{Q}/\mathbb{Z}$ are cosets of the form $\frac{p}{q} + \mathbb{Z}$ and all have finite order (since $q(\frac{p}{q}) \in \mathbb{Z}$). Since isomorphisms map elements of finite order to elements of finite order (Thm 4.6 of Chapter 5, HB p29) all elements in the image of $\mathbb{Q}/\mathbb{Z}$ must have finite order and so nothing can map to $r \in \mathbb{R}$.

*A*lternatively Since $\mathbb{Q}$ is countable (Table, HB p87), we must have countably many cosets in $\mathbb{Q}/\mathbb{Z}$. $\mathbb{R}$ is uncountable (Table, HB p87) and each coset in $\mathbb{R}/\mathbb{Q}$ is countable, and since the union of countably many sets is countable (Table, HB p87) we must have uncountably many cosets in $\mathbb{R}/\mathbb{Q}$ and so this cannot be isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Question 8

(a) Let p be an odd prime and let $R = \mathbb{Z}[\sqrt{-2p}]$, where

$$\mathbb{Z}[\sqrt{-2p}] = \left\{ a + b\sqrt{-2p} : a, b \in \mathbb{Z} \right\}.$$

Define a norm, N, on R by $N(a + b\sqrt{-2p}) = a^2 + 2pb^2$. You may assume that N is multiplicative and that any unit has a norm of 1.

(i) Determine all the units of R.

Solution

If $a + b\sqrt{-2p}$ is a unit then by assumption we have $a^2 + 2pb^2 = 1$. Since $2p > 1$ we must have $b = 0$ and so the only possible units are ± 1. Both of these clearly are units.

(ii) Determine all elements of R with norm equal to $2p$.

Solution

If $N(a + b\sqrt{-2p}) = 2p$ then we have $a^2 + 2pb^2 = 2p$. Since p is odd, $2p$ cannot be a perfect square. Therefore b cannot be zero. If $b > 1$ then the norm is greater than $2p$ so the only solutions are $b = \pm 1$ and the only elements with norm $2p$ are $\pm\sqrt{-2p}$.

[7]

(iii) Prove that the ideal $I = \langle 4, 2 + 2\sqrt{-2p} \rangle$ is not equal to R.

Solution

Elements of I are of the form
$(a + b\sqrt{-2p}) \cdot 4 + (c + d\sqrt{-2p}) \cdot (2 + 2\sqrt{-2p})$. Multiplying out gives:
$4a + 4b\sqrt{-2p} + 2c + 2d\sqrt{-2p} + 2c\sqrt{-2p} - 4dp = 2(2a + c - 2pd) + 2(2b + c + d)\sqrt{-2p}$. Therefore I only contains terms $r + s\sqrt{-2p}$ where r and s are even and so cannot be all of R.

(b) Let α be a root of $x^2 - x - 18$ and let $R = \mathbb{Z}[\alpha]$. You may assume that R is an integral domain, that each element of R can be written as $a + b\alpha$ and that the function $N : R - \{0\} \to \mathbb{N} \cup \{0\}$ given by :

$$N(a + b\alpha) = |a^2 + ab - 18b^2|$$

is a multiplicative and Euclidean norm.

(i) Show that $(943 + 250\alpha)^n$ is a unit in R for any $n \in \mathbb{N}$. *[3]*

Solution

Since R is a Euclidean domain, by Lemma 3.5(b) $r \in R$ is a unit if, and only if, $N(r) = N(1)$. We have:
$N(1) = |1^2 + 1 \cdot 0 - 18 \cdot 0^2| = 1$.
Also $N((943 + 250\alpha) = |943^2 + 943 \cdot 250 - 18 \cdot 250^2| = 1$.
Therefore $943 + 250\alpha$ is a unit. Furthermore, since N is multiplicative, we have $N((943 + 250\alpha)^n) = (N(943 + 250\alpha))^n = 1^n = 1$ and so $(943 + 250\alpha)^n$ is a unit for all $n \in \mathbb{N}$.

[3]

(ii) Explain why every ideal in R is equal to $\langle r \rangle$ for some $r \in R$. *[1]*

Solution

Since R is a Euclidean domain, it is a PID by Cor 2.11 of Chapter 17, HB p95 and so all ideals are principal and so equal to $\langle r \rangle$ for some $r \in R$

[1]

(iii) Let $I = \langle 1 + \alpha \rangle$. Prove that $I = \{a + b\alpha : a \equiv b \pmod{16}, a, b \in \mathbb{Z}\}$. [6]

Solution

First we show that an arbitrary element of I, say $(p + q\alpha)(1 + \alpha)$, can be written as $a + b\alpha : a \equiv b \pmod{16}$ for some a, b in $\mathbb{Z}$.

So, let $(p + q\alpha)(1 + \alpha)$ be an arbitrary element of I. Expanding we have:

$$(p + q\alpha)(1 + \alpha) = a + a\alpha + b\alpha + b\alpha^2.$$

Replacing α^2 by $\alpha + 18$ (since α is a root of $x^2 - x - 18$) gives:

$$(p + q\alpha)(1 + \alpha) = (a + 18b) + (a + 2b)\alpha = r + s\alpha$$

where $r = a + 18b$ and $s = a + 2b$. Therefore $r - s = 16b$ and, by definition, $r \equiv s \pmod{16}$ and $I \subseteq \{a + b\alpha : a \equiv b \pmod{16}\}$.

For the reverse inclusion, let $(x + 16k) + x\alpha$, where $k \in \mathbb{Z}$ be an arbitrary element of $\{a + b\alpha : a \equiv b \pmod{16}\}$.

RTP that we can find an a and b such that

$$(x + 16k) + x\alpha = (a + b\alpha)(1 + \alpha).$$

Note that:

$$a + b\alpha)(1 + \alpha) = (a + 18b) + (a + 2b)\alpha$$

so we need an a and b such that

$$(x + 16k) + x\alpha = a + 18b + (a + 2b)\alpha.$$

Comparing coefficients gives the equations:

$x + 16k = a + 18b$ (1) and $x = a + 2b$ (2). Subtracting (2) from (1) and substituting gives the unique solution $b = k$ and $a = x - 2k$. Therefore $I \supseteq \{a + b\alpha : a \equiv b \pmod{16}\}$ and the result is proved.

[6]

Question 9

(a) Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + 2|x_2 - y_2|.$$

(i) Prove that d is a metric on $\mathbb{R}^2$ by showing that:

i. (M1) holds [3]

ii. (M2) holds [1]

iii. (M3) holds. [3]

(ii) Find the d-closure of $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ in $\mathbb{R}^2$. [3]

(b) Let e denote the induced Euclidean metric on the interval $[0, 1]$, and let $f\colon [0, 1] \to [0, 1]$ be given by

$$f(x) = \frac{1}{5}x^4 + \frac{3}{4}.$$

(i) Verify that for each $x \in [0, 1]$, $0 \leq f(x) \leq 1$. [1]

(ii) Show that f is a contraction for e.
Hint: you may find it helpful to recall that

$$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2).$$

[2]

(iii) Show that $([0, 1], e)$ is a complete metric space. [2]

(iv) Hence, or otherwise, show that f has a unique fixed point in the interval $[0, 1]$. [2]

Solution

(a) (i) i. For (M1): Certainly $d((x_1, x_2), (y_1, y_2)) \geq 0$ Note that

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) = 0 &\text{ if, and only if,} \\ |x_1 - y_1| + 2|x_2 - y_2| = 0 &\text{ if, and only, if} \\ |x_1 - y_1| = 0 \text{ and } 2|x_2 - y_2| &= 0 \end{aligned}$$

which occurs if, and only if, $x_1 = y_1$ and $x_2 = y_2$. That is, if and only if, $(x_1, x_2) = (y_1, y_2)$. 3

ii. This follows from the fact that (M2) hold for modulus:

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= |x_1 - y_1| + 2|x_2 - y_2| \\ &= |y_1 - x_1| + 2|y_2 - x_2| \\ &= d((y_1, y_2), (x_1, x_2)) \end{aligned}$$

1

iii. This follows from the fact that modulus satisfies the triangle inequality: If $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$, then

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= |x_1 - y_1| + 2|x_2 - y_2| \\ &= |x_1 - z_1 + z_1 - y_1| + 2|x_2 - z_2 + z_2 - y_2| \\ &\leq (|x_1 - z_1| + |z_1 - y_1|) + 2(|x_2 - z_2| + |z_2 - y_2|) \\ &= (|x_1 - z_1| + 2|x_2 - z_2|) + (|z_1 - y_1| + 2|z_2 - y_2|) \\ &= d((x_1, x_2), (z_1, z_2)) + d((z_1, z_2), (y_1, y_2)). \end{aligned}$$

3

(ii) Each point of the form $(a, 0)$ is in $\mathrm{Cl}_{(\mathbb{R}^2,d)}(A)$, since the sequence (a_n), where $a_n = (a, 1/n)$ for $n \in \mathbb{N}$, is in A and $d(a_n, (a, 0)) = 2/n$, meaning that $(d(a_n, (a, 0)))$ is a real null sequence. 1

If a point is of the form (a, b), where $b < 0$, then if $(x_1, y_1) \in A$, we estimate

$$d((a, b), (x_1, y_1))) \geq 2|b - y_1| \geq 2|b|.$$

Hence no sequence of points from A can d-converge to points (a, b) with $b < 0$. 1

We conclude that 1

$$\mathrm{Cl}_{(\mathbb{R}^2,d)} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}.$$

(b) (i) For each $x \in [0, 1]$, is certainly the case that $f(x) \geq \frac{3}{4} \geq 0$ and $f(x) = \frac{1}{5}x^4 + \frac{3}{4} \leq \frac{1}{5} + \frac{3}{4} \leq 1$. 1

(ii) For each $x, y \in [0, 1]$,

$$\begin{aligned} d(f(x), f(y)) &= \left|\left(\frac{1}{5}x^4 + \frac{3}{4}\right) - \left(\frac{1}{5}y^4 + \frac{3}{4}\right)\right| \\ &= \frac{1}{5}|x^4 - y^4| \\ &= \frac{1}{5}|x - y| \times |x + y| \times |x^2 + y^2| \\ &\leq \frac{4}{5}|x - y| \\ &= \frac{4}{5}e(x, y). \end{aligned}$$

Hence f is a contraction for e (with a contraction ratio $\frac{4}{5}$). 2

(iii) $[0, 1]$ is a closed subset of $\mathbb{R}$ for the Euclidean metric and $(\mathbb{R}, d^{(1)})$ is a complete metric space. Hence by Proposition 2.2(2) (HB p136), $([0, 1], e)$ is a complete metric space. 2

(iv) f is a contraction mapping on a complete space and so by the Contraction mapping Theorem (HB p138, Theorem 3.6), it has a unique fixed point in the interval $[0, 1]$. 2

[END OF QUESTION PAPER]